J. Fluid Mech. (1965), *vol.* 23, *part* 4, *pp.* 779–786
Printed in Great Britain

Magnetogasdynamic deflagration under the Chapman–Jouguet condition

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(Received 26 February 1965)

An investigation is made into the propagation of a one-dimensional combustion wave, which consists of a flame front and a precursor shock wave which pass down a tube closed at one end, in the presence of a transverse magnetic field in the undisturbed gas at rest. The shock wave is assumed to be of sufficient strength to ionize completely the initially non-electrically-conducting gas and the conditions at the flame front are taken to satisfy the Chapman–Jouguet condition. Details of the solution are compared with the corresponding results for ordinary gasdynamic deflagration.

1. Introduction

The properties of combustion waves in non-ionized gases are fairly well understood, and it is commonly accepted that a steady-state model of the phenomena may be formed of a shock wave followed by a flame front at which exothermal energy is released. Since, however, ionization of a gas may occur at temperatures of the order of 10^4 deg K it is possible that the shock wave may be sufficiently strong to ionize the gas, in which case the description of the flow downstream takes on a magnetogasdynamic aspect. In virtually all the models of such gas-ionizing shock waves with a discontinuity of electrical conductivity σ across them so far studied, an idealized situation has been envisaged in which σ jumps from zero ahead to infinitely large values behind, so that the theory may be expected to be relevant only to those flows with an appropriate magnetic Reynolds number ahead of and behind the front very much less and greater than unity, respectively.

In a previous paper Helliwell (1963) has investigated two particular processes of combustion. It was first shown that the magnetogasdynamic analogue of the Chapman–Jouguet condition is that the velocity of the burnt fluid particles relative to the flame front is equal to the magnetoacoustic speed in the products of combustion. This was followed by a detailed analysis of a steady magnetogasdynamic detonation wave under this condition. The second problem analysed was that of a steady deflagration, with flame front and precursor shock wave moving down a tube closed at one end, under the influence of an external transverse magnetic field, thereby extending the earlier results of Adams & Pack (1959), for ordinary gasdynamic deflagrations, to the magnetogasdynamic régime. The analysis was carried out for a range of values of the shock speed and associated flame speed, and almost all the features of ordinary gasdynamic deflagration were found to be basically unaltered by magnetogasdynamic effects. In particular it was noted that the attainment of the Chapman–Jouguet speed by the flame does not lead to conditions equivalent to those of a detonation wave. The states corresponding to shock speeds with flame speed exceeding the magneto-acoustic speed relative to the burnt gas particles correspond to the magnetogasdynamic analogue of what Courant & Friedrichs (1948) term 'strong deflagrations' and it has been argued that these have no physical import. The consequence of the arguments is that for such faster shocks, as the reaction in the flame proceeds, a position is reached at which the reaction is complete and the flow parameters take up values associated with the Chapman–Jouguet condition. The outcome of this is that the particle speed immediately behind the flame is no longer zero and a rarefaction wave, centred at the closed end, must exist in the burnt gases in order to reduce the burnt particle speed to zero at this end. In this paper the properties of a deflagration are discussed under these conditions.

2. A model deflagration

A one-dimensional model is considered which consists of the propagation, from the closed end of a tube, of a gas-ionizing shock wave across which the electrical conductivity jumps from $\sigma = 0$ to $\sigma = \infty$ followed by a flame front at which exothermal energy is released and across which the Chapman–Jouguet condition holds. Immediately behind the flame there exists a rarefaction wave by passage through which the burnt particles are brought to rest at the closed end of the tube. Upstream of the shock wave, in the undisturbed gas, electric and magnetic fields may exist with mutually orthogonal components parallel to the wave fronts.

The conservation equations for magnetogasdynamic shock waves and flame fronts are well established. For the model described above they lead to two systems of equations which have been presented as equations (12), (13), (14) and (15) of the earlier paper (Helliwell 1963); they are thus not repeated here. Indeed this investigation should be regarded as an extension of the work reported in that paper, in which will be found definition of the notation employed throughout the present study. A schematic picture of the model, together with an associated wave diagram, is shown in figure 1.

The following remarks concerning the validity of the governing equations are appropriate. The exothermal energy Q per unit mass released at the flame front is supposed constant and the absorption of ionization energy at the shock wave is neglected. Whilst the high degree of ionization associated with a jump of conductivity from zero to infinity is unlikely to be consistent with the latter assumption, yet in this paper the jump should be regarded as a scale effect rather than a consequence of full ionization of the initially non-conducting gas. In this case the neglect of the absorption of ionization energy is not unreasonable.

The analysis is made under the further supposition that the deflagration propagates into a perfect gas at rest, and that throughout all transitions the gas remains perfect with uniform polytropic index. For additional simplicity considerations are also restricted to the case when the upstream electric field is absent, in which case the electric field is zero relative to the gas particles upstream of the shock wave and the model becomes formally identical with a pure magnetogasdynamic deflagration in which the initially undisturbed gas is already perfectly conducting.

The calculation of the transition through the shock wave follows exactly as in the previous paper. However, the solution for the jump relations across the flame

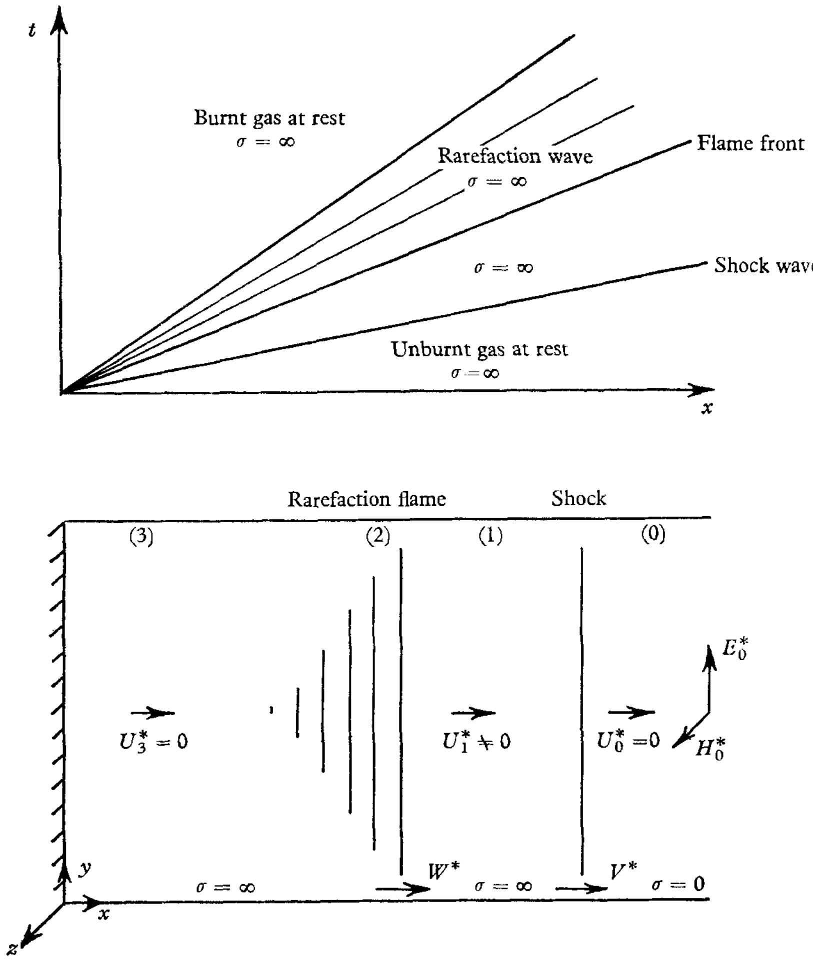

FIGURE 1. Model deflagration under Chapman–Jouguet conditions.

front requires a somewhat different analysis since Chapman–Jouguet conditions are now to be satisfied here. It is found most convenient to solve first for the density ratio ρ_2/ρ_1. One obtains the quartic equation

$$h_5(\rho_2/\rho_1)^4 + h_4(\rho_2/\rho_1)^3 + h_3(\rho_2/\rho_1)^2 + h_2(\rho_2/\rho_1) + h_1 = 0, \tag{1}$$

where
$$h_1 = \{(\gamma+1)/(\gamma-1)\}\{a_1^2/\gamma + \tfrac{1}{2}\alpha^2\rho_1/\rho_0\},$$
$$h_2 = -\{2(\gamma+1)/\gamma\}\{a_1^2/(\gamma-1) + \alpha^2\rho_1/\rho_0 + q\},$$
$$h_3 = \{3(\gamma^2-\gamma-1)/\gamma(\gamma-1)\}\,\alpha^2\rho_1/\rho_0 + \{(\gamma+1)/\gamma(\gamma-1)\}\,a_1^2 + 2q,$$
$$h_4 = \{2(2-\gamma)/(\gamma-1)\}\alpha^2\rho_1/\rho_0,$$
$$h_5 = -\{\tfrac{1}{2}(2-\gamma)/\gamma\}\,\alpha^2\rho_1/\rho_0.$$

Here α is the ratio of Alfvén speed to the ordinary sound speed a_0 upstream of the fronts; $q = Q/a_0^2$. Now it is known that across the flame front in a deflagration the density must fall, so that, in the present problem, $\rho_2/\rho_1 < 1$. It is a simple matter to rearrange equation (1) as a quartic equation in $(1-\rho_2/\rho_1)$ and to see that only one positive root of this equation can exist. Hence there is a single positive root of equation (1) which is less than unity. Consequently the smallest positive root of equation (1) is the required solution. The flame speed and downstream field variables are then obtained without difficulty from equations (19) of the earlier paper.

It has been pointed out that, in order to reduce the speed of the products of combustion to rest at the closed end of the tube, a rarefaction wave must exist in the burnt gas in which the electrical conductivity is infinite. An analysis of one-dimensional simple waves in a perfect infinitely conducting gas in the presence of a transverse magnetic field has recently been carried out by Gundersen (1962). The solution is reduced to quadratures which are integrable in terms of elementary functions if $\gamma = \frac{5}{3}$, a value appropriate to a simple ionized gas, and which will be chosen as the value of γ in the subsequent calculations. Using Gundersen's results one finds that the speed of the wave at any point in the burnt gas is (U^*+C) and thus it is seen that immediately behind the flame front the speed of the wave and flame front are equal. Hence, as in ordinary gasdynamic deflagration theory, the head of the rarefaction wave moves with the flame front.

The additional property governing the behaviour of the simple wave in the case when $\gamma = \frac{5}{3}$ may be written

$$\tfrac{1}{2}U^* - C^3/\beta^2 = \text{const.},$$

where β is the local dimensional Alfvén speed. The conditions at the end of the tube in region (3) behind the rarefaction wave are obtained immediately from this equation. One finds that, since $U_3^* = 0$, the pressure ratio across the wave and the magnetoacoustic speed at the end of the tube are given respectively by

$$p_3/p_2 = \{(c_2^2/\alpha^2)(\tau_2/\tau_0)[1-\tfrac{1}{2}(\alpha^2 u_2^*/c_2^3)(\tau_0/\tau_2)]^{\frac{2}{3}} - (a_2^2/\alpha^2)(\tau_2/\tau_0)\}^5,$$

$$c_3 = (c_2/\alpha^2)(\tau_2/\tau_0)\{c_2^2[1-\tfrac{1}{2}(\alpha^2 u_2^*/c_2^3)(\tau_0/\tau_2)]^{\frac{2}{3}} - a_2^2\} \times \{1-\tfrac{1}{2}(\alpha^2 u_2^*/c_2^3)(\tau_0/\tau_2)\}^{\frac{1}{3}}.$$

Finally the density ratio, temperature ratio and ratio of magnetic field strength across the wave are given by

$$H_3^*/H_2^* = \rho_3/\rho_2 = (T_3/T_2)^{\frac{3}{2}} = (p_3/p_2)^{\frac{3}{5}},$$

whilst in the absence of infinitely large currents, since the gas particles are at rest behind the wave, the electric field E_3^* is zero.

3. Numerical analysis and discussion

Since analytic solution of the relevant equations is not possible a direct numerical analysis has been made and the results presented graphically. Throughout, a value $\gamma = \frac{5}{3}$ has been chosen and the calculations carried out for values of the magnetic field parameter $\alpha^2 = 0$, 100, 500 and of the exothermal energy parameter $q = 50$, 100, 500. The ranges of these parameters are the same as those used in the calculations of the associated model (Helliwell 1963) in which the speed of the precursor shock is lower and Chapman–Jouguet conditions are not reached across the flame front. Here we recall that the values of α^2 correspond to a range of magnetic field strength of order $0 \leqslant H_0^* \leqslant 5000\,\mathrm{G}$ in a gas at atmospheric pressure and higher values at higher pressures. The value $q = 50$ is that for a conventional explosive releasing about 1700 cal/g in a gas at atmospheric pressure and density 1 g/l, whilst $q = 500$ is a value to be associated with thermonuclear fusion in a more diffuse gas.

The transition across the shock wave for higher values of shock speed is similar to that described in the paper just cited, and is therefore not discussed further. The details of the flame front and rarefaction wave are presented in figures 2–7. For completeness these figures show the solution for all shock speeds, the details for the lower values of v^* and pre-Chapman–Jouguet conditions having been transcribed from figures 7–12 of the paper referred to above. It will be recalled that, for a shock wave to exist, $v^* \geqslant \surd(1-\alpha^2)$. As v^* increases from this value a rarefaction wave behind the flame front is not called for until a value of $v^*(=v_C^*)$ is reached such that Chapman–Jouguet conditions are just attained at the flame. The value of v_C^* is clearly indicated in figures 2–7 by the position of the branch point on the various curves. Further each graph is terminated when the shock speed $v^*(=v_D^*)$ equals the flame speed w^* so that conditions equivalent to a steady detonation are reached.

From figures 2–4 it is observed that v_C^* increases with α^2 and so also does v_D^*. This is associated with the increase of $v_{\min}^*$ with α^2. However, although the shock speeds are substantially greater, it is noted that v_C^* and v_D^* are reached over shorter ranges of shock speed as α^2 increases. Increase of q tends to increase these ranges. The flame speed has an almost linear variation with the shock speed. Similar characteristics appear for all values of q. For increasing q, v_C^* and v_D^* occur at progressively higher values of flame speed. However, for large exothermal energy release, the electromagnetic effects are considerably reduced. The magnetoacoustic speed immediately behind the flame front bears an almost linear relationship to the shock speed, whilst that behind the rarefaction wave is sensibly constant except for low values of α^2, when a slight increase with shock velocity occurs.

Consider next the density ratio (which is equal to the ratio of the magnetic field) across the flame and rarefaction wave as shown in figures 5–7. Immediately behind the flame these increase almost linearly with shock speed to the value associated with $v^* = v_D^*$. This final value decreases with increasing α^2 and is always greater than unity. This is one of the few results that may be verified analytically: setting $w^* = v^*$ one finds $u_2^* = v^*(1-\rho_0/\rho_2)$ and thus, since $u_2^* > 0$, it follows that $\rho_2/\rho_0 > 1$. This result also makes it clear that, as u_2^* increases, the density ratio also increases. On the other hand, the density ratio at the closed end of the tube tends, with increase of shock speed, to a constant value which increases slightly with α^2. Consequently the density ratio across the rarefaction wave decreases as α^2 increases.

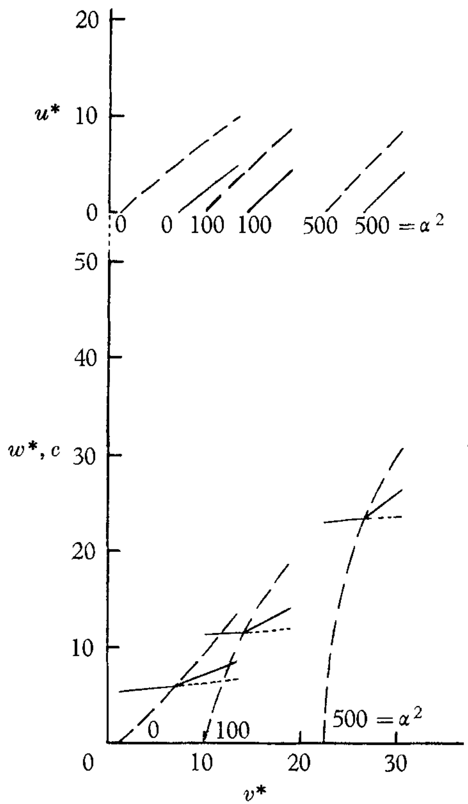

FIGURE 2. Flame, magnetoacoustic and particle speeds ($q = 50$).

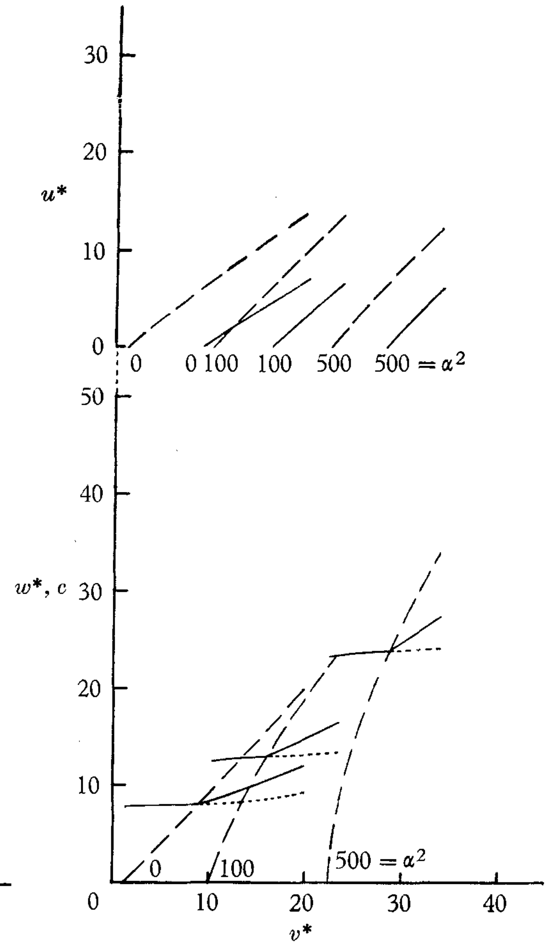

FIGURE 3. Flame, magnetoacoustic and particle speeds ($q = 100$).

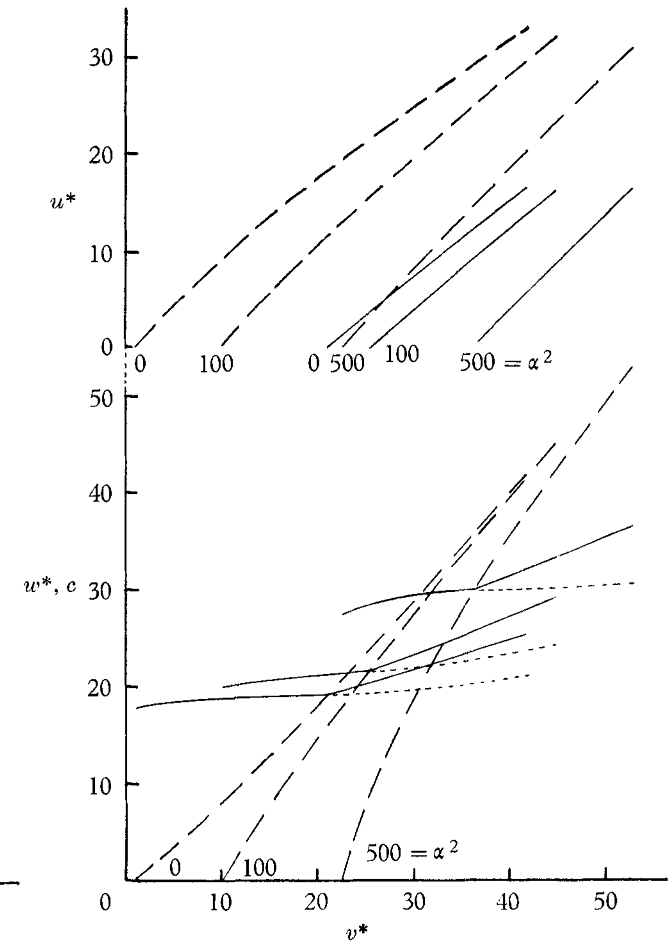

FIGURE 4. Flame, magnetoacoustic and particle speeds ($q = 500$).

Key to FIGURES 2–4. — — —, Flame speed w^*. Particle speed u^*: — — —, ahead of flame; ———, behind flame. Magnetoacoustic speed c: ———, behind flame; ---------, behind rarefaction.

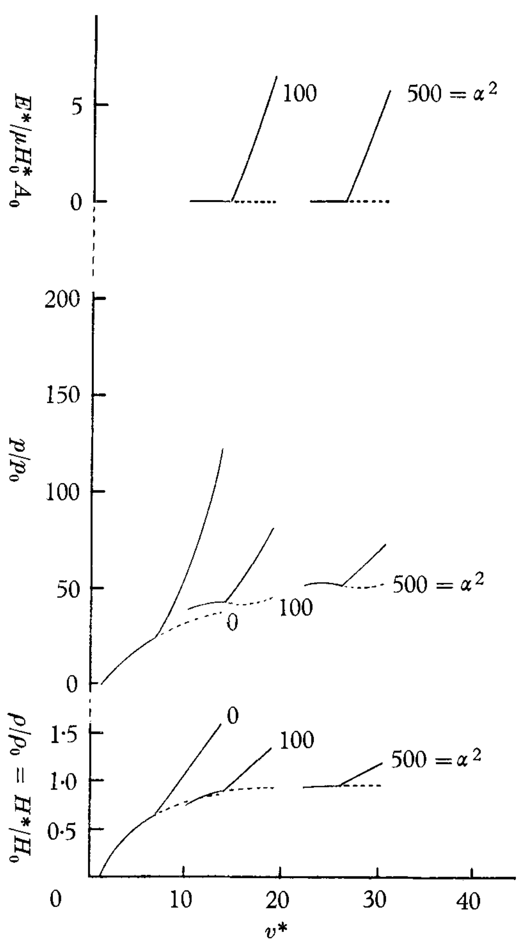

FIGURE 5. Burnt gas state ($q = 50$).

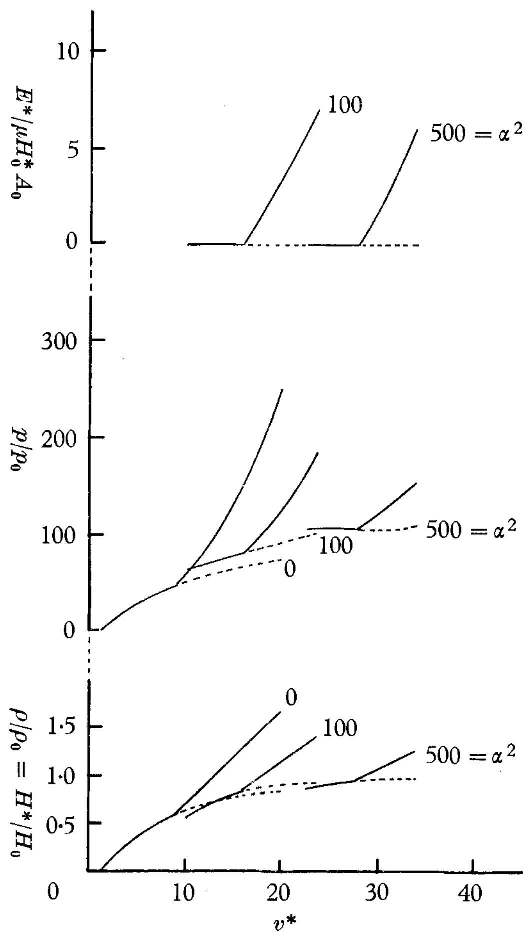

FIGURE 6. Burnt gas state ($q = 100$).

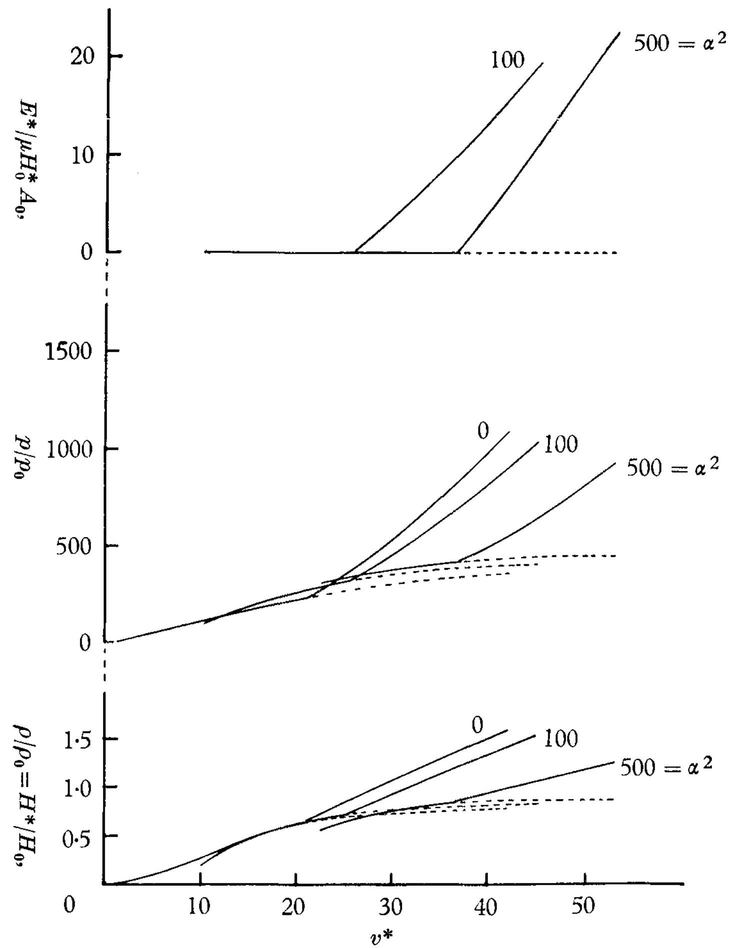

FIGURE 7. Burnt gas state ($q = 500$).

Key to FIGURES 5–7. ———, Behind flame; --------, behind rarefaction.

Figures 5–7 also show that the pressure behind the flame undergoes large changes for comparatively small changes of shock and flame speed. However, as with the density, these changes decrease with increase of α^2, and the pressure when $v^* = v_D^*$ also decreases with increase of α^2. The pressure behind the rarefaction wave shows a decrease to a minimum (for larger values of α^2) before increasing to the final value.

The particle speeds are seen from figures 2–4 to be approximately proportional to the shock speed above the minimum. They are also approximately independent of α^2 but increase almost linearly with q. In the gasdynamic analogue of the present study, Adams & Pack (1959) have shown analytically that the particle speed ahead of the flame at $v^* = v_D^*$ is exactly twice its value at $v^* = v_C^*$, which is itself exactly equal to the value of the particle speed behind the flame at $v^* = v_D^*$. It is noted that both results appear true to a high order of approximation in the magnetogasdynamic case and it is conjectured that they hold exactly, although it has not proved possible to verify this as yet without prohibitive algebra.

Finally, the electric fields behind the flame are shown in figures 5–7. They are of an induced nature and directly proportional to the product of the particle speed and density. The electric fields are thus considerably greater ahead of the flame than behind (cf. Helliwell 1963, figure 5). The fields decrease in strength with increasing α^2 due primarily to a decrease in density, since the particle speed changes little with α^2. The electric fields, however, increase with q, owing to increases in both density and particle speed.

In summary, it is noted that many of the characteristics of ordinary gasdynamic combustion are still evident in the magnetogasdynamic case. In general, for given q, there exist higher wave speeds, lower pressures and densities and approximately unchanged particle speeds for increasing strengths of magnetic field. The transition through the v_C^* point shows a discontinuity in the rates of change with v^* of certain parameters behind the flame, but there is no objection to this as the rarefaction wave immediately adjusts matters so that there is a smooth transition in the gas properties at the closed end of the tube.

The research reported in this paper was supported in part by the United States Air Force under Grant No. AF-EOAR-64-6 and monitored by the European Office, Office of Aerospace Research.

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